Kaluza-Klein Formalism of General Spacetimes

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I describe the Kaluza-Klein approach to general relativity of 4-dimensional spacetimes. This approach is based on the (2,2)-fibration of a generic 4-dimensional spacetime, which is viewed as a local product of a (1+1)-dimensional base manifold and a 2-dimensional fibre space. It is shown that the metric coefficients can be decomposed into sets of fields, which transform as a tensor field, gauge fields, and scalar fields with respect to the infinite dimensional group of the diffeomorphisms of the 2-dimensional fibre space. I discuss a few applications of this formalism.

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I. INTRODUCTION

It has been known for some time that there is a curious correspondence between (self-dual) Yang-Mills equations and the (self-dual) Einstein's equations, when the Yang-Mills gauge symmetry is extended to an infinite dimensional symmetry of (volume-preserving) diffeomorphisms of some auxiliary manifold [1]. It is also well-known that the equations of motion of 2-dimensional non-linear sigma models with the target space as the area-preserving diffeomorphism of an auxiliary 2-surface [2–6] are identical to the the self-dual Einstein's equations written in the Plebañski form [7].

These correspondences are most striking for self-dual cases, and indicate an intriguing possibility that we may be able to reconstruct the full Einstein's general relativity from suitable gauge field theories by replacing the usual finite dimensional gauge symmetry with an infinite dimensional group of the diffeomorphisms of some manifold. If we recall that the gauge symmetry of general relativity is the group of the diffeomorphisms of a 4-dimensional spacetime, this seemingly wild speculation is not totally unreasonable. Recently we have shown that such a description is indeed possible, by rewriting the Einstein-Hilbert action of general relativity of generic 4-dimensional spacetimes in the (2,2)-decomposition [8–13]. In this approach, the 4-dimensional spacetime is viewed, at least for a finite range of the spacetime, as a locally fibred manifold that consists of a (1+1)-dimensional base manifold M_{1+1} and a 2-dimensional fibre space N_2.

The Yang-Mills gauge fields, which naturally appear in this Kaluza-Klein setting [14], are defined on the (1+1)-dimensional base manifold M_{1+1}, and turn out to be valued in the Lie algebra of an infinite dimensional group of the diffeomorphisms of the 2-dimensional fibre space N_2 (i.e. diffN_2). This feature is expected to simplify considerably certain issues concerned with the constraints of general relativity. Namely, in Yang-Mills gauge theories, it is well-known that the Gauss-law constraints associated with the Yang-Mills gauge invariance can be made "trivial", if we consider gauge invariant quantities only. Thus, in principle, one might expect that the problem of solving the constraints of general relativity could be made "trivial", at least for some of them, if such a gauge theory description is possible. The purpose of this paper is to show explicitly that our variables transform as a tensor field, gauge fields, and scalar fields with respect to the diffN_2 transformations, and discuss a general spacetime from the 4-dimensional fibre bundle point of view.

This paper is organized as follows. In section II, we shall outline the kinematics of the (2,2)-decomposition of a generic 4-dimensional spacetime, and introduce the Kaluza-Klein (KK) variables *without* assuming any spacetime isometries. In section III, we shall find the transformation properties of the KK variables with respect to the diffN_2 transformations, and introduce the notion of the diffN_2-*covariant* derivatives. In section IV, we shall write down the Einstein-Hilbert action, and finally, we discuss possible applications of this formalism.

II. KINEMATICS

Let us decompose a generic 4-dimensional spacetime of the Lorentzian signature from the KK perspective, in which the spacetime under consideration is viewed as a 4-dimensional fibre bundle, consisting of a (1+1)-dimensional base

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manifold M_{1+1} and a 2-dimensional fibre space N_2. Let the basis vector fields of M_{1+1} and N_2 be $\partial/\partial x^\mu(=\partial_\mu)$ and $\partial/\partial y^a(=\partial_a)$, respectively, where $\mu=0,1$ and $a=2,3$. The horizontal vector fields $\hat{\partial}_\mu$, which are defined to be orthogonal to N_2, can be expressed as linear combinations of ∂_μ and ∂_a,

$$\hat{\partial}_\mu = \partial_\mu - A_\mu{}^a \partial_a, \tag{2.1}$$

where the fields $A_\mu{}^a$ are functions of (x^μ, y^a). Let us denote by $\gamma^{\mu\nu}$ the inverse metric of the horizontal space spanned by $\hat{\partial}_\mu$, and by ϕ^{ab} the inverse metric of N_2, respectively. In the horizontal lift basis which consists of $\{\hat{\partial}_\mu, \partial_a\}$, the metric of the 4-dimensional spacetime can then be written as [15]

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \gamma^{\mu\nu}\Big(\partial_\mu - A_\mu{}^a\partial_a\Big)\otimes\Big(\partial_\nu - A_\nu{}^b\partial_b\Big) + \phi^{ab}\partial_a\otimes\partial_b. \tag{2.2}$$

In the corresponding dual basis $\{dx^\mu, dy^a + A_\mu{}^a dx^\mu\}$, the metric becomes

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu + \phi_{ab}\Big(dy^a + A_\mu{}^a dx^\mu\Big)\Big(dy^b + A_\nu{}^b dx^\nu\Big). \tag{2.3}$$

Formally the above metric looks similar to the "dimensionally reduced" metric in standard KK theories, but in fact it is quite different. In the standard KK reduction certain isometries are usually assumed, and dimensional reduction is made by projection along the directions generated by these isometries [14]. There the fields $A_\mu{}^a$ are identified as the KK gauge fields associated with the *finite* dimensional isometry group. In this paper, we do *not* assume such isometries: nevertheless, it turns out that the KK idea still works, and as we shall show shortly, the fields $A_\mu{}^a$ can be identified as the gauge fields valued in the *infinite* dimensional Lie algebra of the diffN_2 transformations. Moreover, the fields ϕ_{ab} and $\gamma_{\mu\nu}$ transform as a tensor field and scalar fields with respect to the diffN_2 transformations.

III. DIFFEOMORPHISMS AS A LOCAL GAUGE SYMMETRY

A. Finite transformations

Let us find the transformation properties of the fields ϕ_{ab}, $A_\mu{}^a$, and $\gamma_{\mu\nu}$ with respect to the diffN_2 transformations, which are the following coordinate transformations of N_2, while keeping x^μ constant [16],

$$y^{'a} = y^{'a}(x,y), \qquad x^{'\mu} = x^\mu. \tag{3.1}$$

Thus we have

$$dy^a = \frac{\partial y^a}{\partial y^{'c}}\Big\{dy^{'c} - \Big(\frac{\partial y^{'c}}{\partial x^\mu}\Big)dx^{'\mu}\Big\}, \qquad dx^\mu = dx^{'\mu}. \tag{3.2}$$

In the new coordinates the term proportional to $dx^\mu dy^a$ in (2.3) becomes, keeping the (x^μ, y^a) dependence explicit,

$$\begin{aligned}&2\phi_{ab}(x,y)A_\mu{}^a(x,y)dx^\mu dy^b\\ &= 2\Big(\frac{\partial y^a}{\partial y^{'c}}\Big)\Big(\frac{\partial y^b}{\partial y^{'d}}\Big)\phi_{ab}(x,y)\Big(\frac{\partial y^{'d}}{\partial y^e}\Big)A_\mu{}^e(x,y)dx^{'\mu}\Big\{dy^{'c} - \Big(\frac{\partial y^{'c}}{\partial x^\nu}\Big)dx^{'\nu}\Big\},\end{aligned} \tag{3.3}$$

where the identity

$$\Big(\frac{\partial y^a}{\partial y^{'d}}\Big)\Big(\frac{\partial y^{'d}}{\partial y^e}\Big) = \delta_e{}^a \tag{3.4}$$

was used. Also the term proportional to $dy^a dy^b$ becomes

$$\begin{aligned}&\phi_{ab}(x,y)dy^a dy^b\\ &= \Big(\frac{\partial y^a}{\partial y^{'c}}\Big)\Big(\frac{\partial y^b}{\partial y^{'d}}\Big)\phi_{ab}(x,y)\Big\{dy^{'c}dy^{'d} - 2\Big(\frac{\partial y^{'d}}{\partial x^\mu}\Big)dy^{'c}dx^{'\mu} + \Big(\frac{\partial y^{'c}}{\partial x^\mu}\Big)\Big(\frac{\partial y^{'d}}{\partial x^\nu}\Big)dx^{'\mu}dx^{'\nu}\Big\}.\end{aligned} \tag{3.5}$$

After rearranging terms, the metric (2.3) can be written as, in the new coordinates,

$$
\begin{aligned}
ds^2 = &\gamma_{\mu\nu}(x,y)dx^{'\mu}dx^{'\nu} + \Big(\frac{\partial y^a}{\partial y^{'c}}\Big)\Big(\frac{\partial y^b}{\partial y^{'d}}\Big)\phi_{ab}(x,y)dy^{'c}dy^{'d} \\
&+2\Big(\frac{\partial y^a}{\partial y^{'c}}\Big)\Big(\frac{\partial y^b}{\partial y^{'d}}\Big)\phi_{ab}(x,y)\Big\{\Big(\frac{\partial y^{'d}}{\partial y^e}\Big)A_\mu^{\ e}(x,y) - \frac{\partial y^{'d}}{\partial x^\mu}\Big\}dx^{'\mu}dy^{'c} \\
&+\phi_{ab}(x,y)\Big\{A_\mu^{\ a}(x,y)A_\nu^{\ b}(x,y) - 2\Big(\frac{\partial y^a}{\partial y^{'c}}\Big)\Big(\frac{\partial y^b}{\partial y^{'d}}\Big)\Big(\frac{\partial y^{'d}}{\partial y^e}\Big)A_\mu^{\ e}(x,y)\Big(\frac{\partial y^{'c}}{\partial x^\nu}\Big) \\
&+\Big(\frac{\partial y^a}{\partial y^{'c}}\Big)\Big(\frac{\partial y^b}{\partial y^{'d}}\Big)\Big(\frac{\partial y^{'c}}{\partial x^\mu}\Big)\Big(\frac{\partial y^{'d}}{\partial x^\nu}\Big)\Big\}dx^{'\mu}dx^{'\nu},
\end{aligned} \tag{3.6}
$$

which must be equal to

$$
ds^{'2} = \gamma'_{\mu\nu}(x',y')dx^{'\mu}dx^{'\nu} + \phi'_{ab}(x',y')\Big\{dy^{'a} + A_\mu^{'\ a}(x',y')dx^{'\mu}\Big\}\Big\{dy^{'b} + A_\nu^{'\ b}(x',y')dx^{'\nu}\Big\}, \tag{3.7}
$$

since the line element is invariant under the diffN_2 transformations. If we compare terms containing $dy^{'a}dy^{'b}$, we find that $\phi_{ab}(x,y)$ transform as

$$
\phi'_{ab}(x',y') = \Big(\frac{\partial y^c}{\partial y^{'a}}\Big)\Big(\frac{\partial y^d}{\partial y^{'b}}\Big)\phi_{cd}(x,y). \tag{3.8}
$$

This shows that $\phi_{ab}(x,y)$ is a tensor field with respect to the diffN_2 transformations. If we use the equation (3.8) in (3.6), the metric becomes

$$
\begin{aligned}
ds^2 = &\gamma_{\mu\nu}(x,y)dx^{'\mu}dx^{'\nu} + \phi'_{cd}(x',y')dy^{'c}dy^{'d} + 2\phi'_{cd}(x',y')\Big\{\Big(\frac{\partial y^{'d}}{\partial y^a}\Big)A_\mu^{\ a}(x,y) - \frac{\partial y^{'d}}{\partial x^\mu}\Big\}dx^{'\mu}dy^{'c} \\
&+\phi'_{cd}(x',y')\Big\{\Big(\frac{\partial y^{'c}}{\partial y^a}\Big)A_\mu^{\ a}(x,y) - \frac{\partial y^{'c}}{\partial x^\mu}\Big\}\Big\{\Big(\frac{\partial y^{'d}}{\partial y^b}\Big)A_\nu^{\ b}(x,y) - \frac{\partial y^{'d}}{\partial x^\nu}\Big\}dx^{'\mu}dx^{'\nu},
\end{aligned} \tag{3.9}
$$

from which we deduce the following transformation properties of $A_\mu^{\ a}(x,y)$ and $\gamma_{\mu\nu}(x,y)$

$$
A_\mu^{'\ a}(x',y') = \Big(\frac{\partial y^{'a}}{\partial y^b}\Big)A_\mu^{\ b}(x,y) - \frac{\partial y^{'a}}{\partial x^\mu}(x,y), \tag{3.10}
$$

$$
\gamma'_{\mu\nu}(x',y') = \gamma_{\mu\nu}(x,y), \tag{3.11}
$$

under the diffN_2 transformations.

B. Infinitesimal transformations

It will be instructive to examine the infinitesimal transformations corresponding to the above finite diffN_2 transformations. The infinitesimal diffN_2 transformations consist of the following transformations

$$
y^{'a} = y^a + \xi^a(x,y), \qquad x^{'\mu} = x^\mu \qquad (\mathrm{O}(\xi^2) \ll 1), \tag{3.12}
$$

where $\xi^a(x,y)$ is an arbitrary, infinitesimal, function of (x^μ, y^a). From this it follows that

$$
\frac{\partial y^c}{\partial y^{'a}} = \delta_a^{\ c} - \frac{\partial \xi^c}{\partial y^a} + \cdots, \tag{3.13}
$$

where $\cdots$ means terms of $\mathrm{O}(\xi^2)$. If we expand the l.h.s. of the equation (3.8) in ξ^a, it becomes

$$
\phi'_{ab}(x', y+\xi) = \phi'_{ab}(x,y) + \xi^c\frac{\partial}{\partial y^c}\phi_{ab}(x,y) + \cdots, \tag{3.14}
$$

whereas the r.h.s. becomes

$$
\Big(\frac{\partial y^c}{\partial y^{'a}}\Big)\Big(\frac{\partial y^d}{\partial y^{'b}}\Big)\phi_{cd}(x,y) = \phi_{ab}(x,y) - \frac{\partial \xi^c}{\partial y^a}\phi_{cb}(x,y) - \frac{\partial \xi^c}{\partial y^b}\phi_{ac}(x,y) + \cdots. \tag{3.15}
$$

Thus we have

$$
\begin{aligned}
\delta\phi_{ab}(x,y) &\equiv \phi'_{ab}(x,y) - \phi_{ab}(x,y) \\
&= -\xi^c\partial_c\phi_{ab}(x,y) - (\partial_a\xi^c)\phi_{cb}(x,y) - (\partial_b\xi^c)\phi_{ac}(x,y) \\
&= -[\xi,\phi]_{\mathrm{L}ab},
\end{aligned} \tag{3.16}
$$

where the subscript $_\mathrm{L}$ denotes the Lie derivative along the vector field $\xi \equiv \xi^a\partial_a$, i.e.

$$
[\xi,\phi]_{\mathrm{L}ab} = \xi^c\partial_c\phi_{ab} + (\partial_a\xi^c)\phi_{cb} + (\partial_b\xi^c)\phi_{ac}. \tag{3.17}
$$

It is a straightforward exercise to derive the infinitesimal transformation properties $A_\mu{}^a$ and $\gamma_{\mu\nu}$ from (3.10) and (3.11). They are found to be

$$
\begin{aligned}
\delta A_\mu{}^a(x,y) &= -\partial_\mu\xi^a + [A_\mu,\ \xi]_\mathrm{L}^a \\
&= -\partial_\mu\xi^a + A_\mu{}^c\partial_c\xi^a - \xi^c\partial_c A_\mu{}^a, \qquad (3.18)\\
\delta\gamma_{\mu\nu}(x,y) &= -[\xi,\gamma_{\mu\nu}]_\mathrm{L} \\
&= -\xi^a\partial_a\gamma_{\mu\nu}, \qquad (3.19)
\end{aligned}
$$

where $[A_\mu,\ \xi]_\mathrm{L}^a$ and $[\xi,\gamma_{\mu\nu}]_\mathrm{L}$ are the Lie derivatives of ξ^a and $\gamma_{\mu\nu}$ along the vector fields $A_\mu = A_\mu{}^c\partial_c$ and $\xi = \xi^c\partial_c$, respectively. Notice that the Lie derivative acts on the fibre space index (a) only. The equations (3.16), (3.18), and (3.19) clearly show that the metric components $\{\phi_{ab}, A_\mu{}^a, \gamma_{\mu\nu}\}$ transform as a tensor field, gauge fields, and scalar fields under the diffN_2 transformations, respectively.

C. diffN_2-covariant derivative

Using the Lie derivative along the diffN_2-valued gauge fields, the diffN_2-*covariant* derivative D_μ can be *naturally* defined as

$$
D_\mu = \partial_\mu - [A_\mu,\ \ ]_\mathrm{L}. \tag{3.20}
$$

With this definition, the equation (3.18) can be written as

$$
\delta A_\mu{}^a = -D_\mu\xi^a, \tag{3.21}
$$

which suggests that the diffN_2-valued field strength $F_{\mu\nu}{}^a$ be defined as

$$
[D_\mu, D_\nu]_\mathrm{L}\eta = -F_{\mu\nu}{}^a\partial_a\eta \tag{3.22}
$$

for an arbitrary scalar function η, where $F_{\mu\nu}{}^a$ is given by

$$
\begin{aligned}
F_{\mu\nu}{}^a &= \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a - [A_\mu, A_\nu]_\mathrm{L}^a \\
&= \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a - A_\mu{}^c\partial_c A_\nu{}^a + A_\nu{}^c\partial_c A_\mu{}^a.
\end{aligned} \tag{3.23}
$$

Similarly, the diffN_2-covariant derivative of ϕ_{ab} is defined as

$$
\begin{aligned}
D_\mu\phi_{ab} &= \partial_\mu\phi_{ab} - [A_\mu,\phi]_{\mathrm{L}ab} \\
&= \partial_\mu\phi_{ab} - A_\mu{}^c\partial_c\phi_{ab} - (\partial_a A_\mu{}^c)\phi_{bc} - (\partial_b A_\mu{}^c)\phi_{ac}.
\end{aligned} \tag{3.24}
$$

It remains to show that $F_{\mu\nu}{}^a$ and $D_\mu\phi_{ab}$ transform *covariantly* under the infinitesimal diffN_2 transformations (3.12). Let us consider $D_\mu\phi_{ab}$ first. The infinitesimal transformation of $D_\mu\phi_{ab}$ becomes

$$
\delta(D_\mu\phi_{ab}) = -\partial_\mu\Big([\xi,\phi]_{\mathrm{L}ab}\Big) + [A_\mu,[\xi,\phi]_\mathrm{L}]_{\mathrm{L}ab} + [D_\mu\xi,\phi]_{\mathrm{L}ab}, \tag{3.25}
$$

where we used the equations (3.16) and (3.18), and the Lie brackets are

$$
[A_\mu,[\xi,\phi]_\mathrm{L}]_{\mathrm{L}ab} = A_\mu{}^c\partial_c\Big([\xi,\phi]_{\mathrm{L}ab}\Big) + (\partial_a A_\mu{}^c)[\xi,\phi]_{\mathrm{L}bc} + (\partial_b A_\mu{}^c)[\xi,\phi]_{\mathrm{L}ac}, \tag{3.26}
$$

$$
[D_\mu\xi,\phi]_{\mathrm{L}ab} = (D_\mu\xi^c)(\partial_c\phi_{ab}) + \partial_a(D_\mu\xi^c)\phi_{bc} + \partial_b(D_\mu\xi^c)\phi_{ac}. \tag{3.27}
$$

Using the Leibniz rule of the derivative ∂_μ

$$\partial_\mu\Big([\xi,\phi]_{\mathrm{L}ab}\Big) = [\partial_\mu\xi,\phi]_{\mathrm{L}ab} + [\xi,\partial_\mu\phi]_{\mathrm{L}ab}, \tag{3.28}$$

and the properties of the Lie bracket

$$[D_\mu\xi,\phi]_{\mathrm{L}ab} = [\partial_\mu\xi,\phi]_{\mathrm{L}ab} - [[A_\mu,\xi]_{\mathrm{L}},\phi]_{\mathrm{L}ab}, \tag{3.29}$$

$$[A_\mu,[\xi,\phi]_{\mathrm{L}}]_{\mathrm{L}ab} = -[\xi,[\phi,A_\mu]_{\mathrm{L}}]_{\mathrm{L}ab} - [\phi,[A_\mu,\xi]_{\mathrm{L}}]_{\mathrm{L}ab}, \tag{3.30}$$

we find that the equation (3.25) becomes

$$\begin{aligned}\delta(D_\mu\phi_{ab}) &= -[\xi,\partial_\mu\phi]_{\mathrm{L}ab} + [\xi,[A_\mu,\phi]_{\mathrm{L}}]_{\mathrm{L}ab}\\ &= -[\xi,D_\mu\phi]_{\mathrm{L}ab},\end{aligned} \tag{3.31}$$

which shows that $D_\mu\phi_{ab}$ transforms *covariantly* under the diffN_2 transformation.

Similarly, the infinitesimal transformation $\delta F_{\mu\nu}{}^a$ becomes

$$\delta F_{\mu\nu}{}^a = \partial_\mu\Big([A_\nu,\xi]^a_{\mathrm{L}}\Big) + [D_\mu\xi,A_\nu]^a_{\mathrm{L}} - (\mu\leftrightarrow\nu). \tag{3.32}$$

Using the following identities

$$\partial_\mu\Big([A_\nu,\xi]^a_{\mathrm{L}}\Big) = [\partial_\mu A_\nu,\xi]^a_{\mathrm{L}} + [A_\nu,\partial_\mu\xi]^a_{\mathrm{L}}, \tag{3.33}$$

$$[D_\mu\xi,A_\nu]^a_{\mathrm{L}} = -[A_\nu,D_\mu\xi]^a_{\mathrm{L}} = -[A_\nu,\partial_\mu\xi]^a_{\mathrm{L}} + [A_\nu,[A_\mu,\xi]_{\mathrm{L}}]^a_{\mathrm{L}}, \tag{3.34}$$

we find that

$$\begin{aligned}\delta F_{\mu\nu}{}^a &= [\partial_\mu A_\nu - \partial_\nu A_\mu,\xi]^a_{\mathrm{L}} + [A_\nu,[A_\mu,\xi]_{\mathrm{L}}]^a_{\mathrm{L}} - [A_\mu,[A_\nu,\xi]_{\mathrm{L}}]^a_{\mathrm{L}}\\ &= -[\xi,F_{\mu\nu}]^a_{\mathrm{L}},\end{aligned} \tag{3.35}$$

where we used the Jacobi identity

$$[A_\nu,[A_\mu,\xi]_{\mathrm{L}}]^a_{\mathrm{L}} = -[A_\mu,[\xi,A_\nu]_{\mathrm{L}}]^a_{\mathrm{L}} - [\xi,[A_\nu,A_\mu]_{\mathrm{L}}]^a_{\mathrm{L}}. \tag{3.36}$$

Therefore it follows that

$$\delta F_{\mu\nu}{}^a = -[\xi,F_{\mu\nu}]^a_{\mathrm{L}}, \tag{3.37}$$

which shows that $F_{\mu\nu}{}^a$ is indeed the diffN_2-valued field strength.

It must be marked here that, in the (2,2)-KK formalism, the Lie derivative, rather than the covariant derivative, appears naturally. The appearance of an infinite dimensional symmetry such as diffN_2 is not surprising, since in general relativity the underlying gauge symmetry is the infinite dimensional group of the diffeomorphisms of a 4-dimensional spacetime. The point is that it is the diffN_2 symmetry, the subgroup of the diffeomorphisms of a 4-dimensional spacetime, that shows up as a local gauge symmetry of the Yang-Mills type. This implies that the (2,2)-KK formalism can be made a viable method of studying general relativity from the standpoint of the (1+1)-dimensional Yang-Mills gauge theory with the diffN_2 symmetry as a local gauge symmetry.

IV. THE ACTION

The Einstein-Hilbert action in this KK formalism is given by

$$\begin{aligned}I = \int d^2x d^2y\,\sqrt{-\gamma}\sqrt{\phi}\,\Big[&\gamma^{\mu\nu}\hat{\mathrm{R}}_{\mu\nu} + \phi^{ac}\mathrm{R}_{ac} + \frac{1}{4}\gamma^{\mu\nu}\gamma^{\alpha\beta}\phi_{ab}F_{\mu\alpha}{}^aF_{\nu\beta}{}^b\\ &+\frac{1}{4}\gamma^{\mu\nu}\phi^{ab}\phi^{cd}\Big\{(D_\mu\phi_{ac})(D_\nu\phi_{bd}) - (D_\mu\phi_{ab})(D_\nu\phi_{cd})\Big\}\\ &+\frac{1}{4}\phi^{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}\Big\{(\partial_a\gamma_{\mu\alpha})(\partial_b\gamma_{\nu\beta}) - (\partial_a\gamma_{\mu\nu})(\partial_b\gamma_{\alpha\beta})\Big\}\Big] + \int d^2xd^2y\,(\partial_AS^A).\end{aligned} \tag{4.1}$$

Let us summarize the notations:
1. The curvature tensors $\hat{\mathrm{R}}_{\mu\nu}$ and R_{ac} are defined as

$$\hat{\mathrm{R}}_{\mu\nu} = \hat{\partial}_\mu \hat{\Gamma}_{\alpha\nu}{}^{\alpha} - \hat{\partial}_\alpha \hat{\Gamma}_{\mu\nu}{}^{\alpha} + \hat{\Gamma}_{\mu\beta}{}^{\alpha} \hat{\Gamma}_{\alpha\nu}{}^{\beta} - \hat{\Gamma}_{\beta\alpha}{}^{\beta} \hat{\Gamma}_{\mu\nu}{}^{\alpha}, \tag{4.2}$$

$$\mathrm{R}_{ac} = \partial_a \Gamma_{bc}{}^{b} - \partial_b \Gamma_{ac}{}^{b} + \Gamma_{ad}{}^{b} \Gamma_{bc}{}^{d} - \Gamma_{db}{}^{d} \Gamma_{ac}{}^{b}, \tag{4.3}$$

$$\hat{\Gamma}_{\mu\nu}{}^{\alpha} = \frac{1}{2} \gamma^{\alpha\beta} \Big(\hat{\partial}_\mu \gamma_{\nu\beta} + \hat{\partial}_\nu \gamma_{\mu\beta} - \hat{\partial}_\beta \gamma_{\mu\nu} \Big), \tag{4.4}$$

$$\Gamma_{ab}{}^{c} = \frac{1}{2} \phi^{cd} \Big(\partial_a \phi_{bd} + \partial_b \phi_{ad} - \partial_d \phi_{ab} \Big). \tag{4.5}$$

2. The last term in (4.1) is a surface integral, where $S^A = (S^\mu, S^a)$ is given by

$$S^\mu = \sqrt{-\gamma} \sqrt{\phi}\, j^\mu, \tag{4.6}$$

$$S^a = \sqrt{-\gamma} \sqrt{\phi} \Big(-A_\mu{}^{a} j^\mu + j^a \Big), \tag{4.7}$$

$$j^\mu = \gamma^{\mu\nu} \phi^{ab} D_\nu \phi_{ab}, \tag{4.8}$$

$$j^a = \phi^{ab} \gamma^{\mu\nu} \partial_b \gamma_{\mu\nu}. \tag{4.9}$$

One can easily recognize that this action is in a form of a (1+1)-dimensional field theory action. In geometrical terms the above action can be understood as follows. $\gamma^{\mu\nu}\hat{\mathrm{R}}_{\mu\nu}$ can be interpreted as the "gauged" scalar curvature of M_{1+1}, since the diffN_2-valued gauge fields are coupled to $\gamma_{\mu\nu}$ and $\hat{\Gamma}_{\mu\nu}{}^{\alpha}$ in the formulae (4.2) and (4.4). $\phi^{ac}\mathrm{R}_{ac}$ is the scalar curvature of N_2, which is proportional to the Euler characteristics χ when integrated over N_2.

The remaining terms in (4.1) are the *extrinsic* terms, telling us how M_{1+1} and N_2 are embedded into the enveloping 4-dimensional spacetime. Each term in (4.1) is manifestly diffN_2-invariant, and the y^a-dependence of each term is completely "hidden" in the Lie derivatives. In this sense we may view the fibre space N_2 as a kind of "internal" space as in Yang-Mills theory. Thus, the Einstein-Hilbert action is describable as a (1+1)-dimensional Yang-Mills type gauge theory interacting with scalar fields and (1+1)-dimensional non-linear sigma fields of generic types, with couplings to curvatures of two 2-surfaces. The associated Yang-Mills gauge symmetry is the diffN_2 symmetry.

V. DISCUSSIONS

In this paper, we presented the KK formalism of general relativity of generic 4-dimensional spacetimes, viewing the spacetime as a local product of the (1+1)-dimensional base manifold and the 2-dimensional fibre space. Within this framework, we made a decomposition of a given 4-dimensional spacetime metric into sets of fields which transform as a tensor field, gauge fields, and scalar fields under the group of the diffeomorphisms on N_2.

In connection with issues of quantum gravity, this KK approach has the following aspects which deserve further remarks. For instance, solving the Einstein's constraint equations or constructing the gauge invariant physical observables is known to be one of the most important problems in quantum general relativity. In our formalism, the diffeomorphisms of the 2-dimensional space N_2 plays the role of a local gauge symmetry *exactly* as in Yang-Mills theory. Therefore the two constraint equations associated with the diffN_2 transformations can be "automatically" solved, using the diffN_2-invariant scalars. However, there are two additional constraint equations which require further studies in order to fully take care of the four Einstein's constraint equations [17].

It should be also stressed that the Lie derivative appears naturally in this formalism, via the *minimal* couplings to the diffN_2-valued gauge fields. In the standard (3+1)-formalism, the natural derivative operator is the metric-compatible covariant derivative, which requires the metric be non-degenerate. The Lie derivative, on the other hand, can be defined even when the metric is degenerate. For instance, at null infinity $\mathcal{I}^+$ of the asymptotically flat spacetimes, the natural derivative operator is the Lie derivative, rather than the covariant derivative, because the metric on $\mathcal{I}^+$ is degenerate with the signature $(0, +, +)$ [18]. Therefore, the KK formalism, based on the notion of the Lie derivative, should be extendable to spacetimes where the metric is degenerate, which would be difficult to describe in conventional approaches.

Finally, it will be a challenging problem to try to *reinterpret* the exact solutions of the Einstein's equations from this gauge theory point of view. This seems very interesting, for there are a number of exact solutions of the Einstein's equations which do not permit sensible physical interpretations from the 4-dimensional spacetime perspective [16].

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[1] L.J. Mason and E.T. Newman, Comm. Math. Phys. 121 (1989) 659.
[2] Q.H. Park, Phys. Lett. B 238 (1990) 287.
[3] Q.H. Park, J. Mod. Phys. A 7 (1992) 1415.
[4] I. Bakas, Phys. Lett. B 228 (1989) 57.
[5] E.G. Floratos, Phys. Lett. B 228 (1989) 335.
[6] D.B. Fairlie, P. Fletcher and C.K. Zachos, J. Math. Phys. 31 (1990) 1088.
[7] J.F. Plebañski, J. Math. Phys. 16 (1975) 2395.
[8] Y.M. Cho, Q.H. Park, K.S. Soh and J.H. Yoon, Phys. Lett. B 286 (1992) 251.
[9] J.H. Yoon, Phys. Lett. B 308 (1993) 240.
[10] J.H. Yoon, in: Directions in General Relativity, ed. B.L. Hu and T.A. Jacobson (Cambridge University Press, 1993).
[11] R.A. d'Inverno and J. Stachel, J. Math. Phys. 19 (1978) 2447.
[12] R.A. d'Inverno and J. Smallwood, Phys. Rev. D 22 (1980) 1233.
[13] S. Hayward, Class. Quantum Grav. 10 (1993) 779.
[14] Y.M. Cho, J. Math. Phys. 16 (1975) 2029.
[15] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation (Freeman, 1972).
[16] D. Kramer *et al*, Exact Solutions of Einstein's Field Equations, ed. E. Schmutzer (Cambridge University Press, 1980).
[17] J.H. Yoon, *in preparation.*
[18] R. Geroch, in: Asymptotic Structure of Space-time, ed. P. Esposito and L. Witten (Plenum, 1977).